

## SINGULAR OPEN BOOK STRUCTURES FROM REAL MAPPINGS

MIHAI TIBĂR, CHEN YING, AND RAIMUNDO N. ARAÚJO DOS SANTOS

ABSTRACT. We prove extensions of Milnor's theorem for germs with nonisolated singularity and use them to find new classes of genuine real analytic mappings  $\psi$  with positive dimensional singular locus  $\text{Sing } \psi \subset \psi^{-1}(0)$ , for which the Milnor fibration exists and yields an open book structure with singular binding.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$  be some analytic mapping germ,  $m \geq p \geq 2$ , and let  $V := \psi^{-1}(0)$ . The special case of  $\text{Sing } \psi = \{0\}$  has been considered by Milnor; his [Mi, Theorem 11.2] tells that there exists a mapping

$$(1) \quad S_\varepsilon^{m-1} \setminus K_\varepsilon \rightarrow S_\delta^{p-1}$$

which is a locally trivial fibration and its diffeomorphism type is independent of  $\varepsilon > 0$  small enough and of  $0 < \delta \ll \varepsilon$ . Moreover, the sphere  $S_\varepsilon^{m-1}$  is endowed with what one may call higher open book structure with binding  $K_\varepsilon$  (cf [AT2, Definition 2.1]). Let us remark that the condition  $V \cap \text{Sing } \psi \subset \{0\}$  is the most general one under which higher open book structures with binding  $K$  may exist. Our investigation focuses on the situation  $\text{Sing } \psi \subset V$ , when the link  $K$  is no more a manifold, but which is the most general situation in which a locally trivial fibration (1) may exist. Moreover, in the real setting the mapping  $\psi$  does not necessarily have a “good” structure near  $V$ ; this is indeed different from the case of a single holomorphic function studied by Milnor (see (2) below), but if we increase the number of holomorphic functions then the behaviour may be again “bad” and one may relate it to the failure of the Thom condition along  $V$ , a well-known phenomenon in the setting of holomorphic mappings. See §4.1 for several remarks about conditions along  $V$ .

**Definition 1.1.** We say that the pair  $(K, \theta)$  is a *higher open book structure with singular binding* on a manifold  $M$  (or a *singular fibered link*) if  $K \subset M$  is a singular real subvariety and  $\theta : M \setminus K \rightarrow S_1^{p-1}$  is a locally trivial smooth fibration such that  $K$  admits a neighbourhood  $N$  for which the restriction  $\theta|_{N \setminus K}$  is the composition  $N \setminus K \xrightarrow{h} B^p \setminus \{0\} \xrightarrow{s/\|s\|} S_1^{p-1}$  where  $h$  is a locally trivial fibration.

---

*Date:* 2nd version, December 20, 2010.

*2000 Mathematics Subject Classification.* 32S55, 14D06, 14P15, 58K05, 57R45.

*Key words and phrases.* singularities of real analytic mappings, open book decompositions, Milnor fibrations.

M.T. acknowledges support from the French *Agence Nationale de la Recherche*, grant ANR-08-JCJC-0118-01.

R.N.A.D.S. acknowledges support from the Brazilian CNPq, PQ-2, proc. num. 305183/2009-5, and FAPESP proc. num. 2009/14383-3.

One says that  $K$  is the *binding* and that the (closures of) the fibers of  $\theta$  are the *pages* of the *open book*.

Let us notice that from the above definition it follows that  $\theta$  is surjective.

Open books are frequent in the literature, they are known under various other names like *fibered links* or *Neuwirth-Stallings pairs* [Lo] or even *Lefschetz pencils*. In the classical definition of an open book (see e.g. [Wi]) we have  $p = 2$ ,  $K \subset M$  is a 2-codimensional submanifold which admits a neighbourhood  $N$  diffeomorphic to  $B^2 \times K$  for which  $K$  is identified with  $\{0\} \times K$  and the restriction  $\theta|_{N \setminus K}$  is the following composition with the natural projections  $N \setminus K \xrightarrow{\text{difeo}} (B^2 \setminus \{0\}) \times K \xrightarrow{\text{proj}} B^2 \setminus \{0\} \rightarrow S^1$ . In [AT2] we have defined a “higher” open book structure, in which  $p \geq 2$ .

Let us point out that the notion of “open book with singular binding” preserves the two main aspects of the classical definitions, namely that the complement of the link fibers over the unit sphere and that the fibration has an especially nice behaviour near the link (“like an open book”) and the latter gives the name of the object itself.

We first formulate the following extension<sup>1</sup> of [Mi, Theorem 11.2] which uses Definition 1.1, the proof of which is derived in §2 only from carefully revisiting Milnor’s construction:

**Theorem 1.2.** [Mi]

*Let  $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$  be an analytic mapping germ with  $\text{Sing } \psi \subset V$ . Suppose that  $\overline{M(\psi) \setminus V} \cap V = \{0\}$ . Then there exists a higher open book structure with singular binding  $(K_\varepsilon, \theta)$  on  $S_\varepsilon^{m-1}$ , which is independent of  $\varepsilon > 0$  small enough, up to isotopies.*

The definition of the set  $M(\psi)$  involves the  $\rho$ -regularity, which expresses the transversality of the fibres of a mapping  $\psi$  to the levels of  $\rho$ . It is a basic tool, used by many authors (Milnor, Mather, Looijenga, Bekka etc) in the local stratified setting as well as at infinity, like in e.g. [NZ, Ti1, Ti2], in order to produce locally trivial fibrations. As in [AT1], in this paper we shall tacitly use as function  $\rho$  the Euclidean distance function and its open balls and spheres of radius  $\varepsilon$ , currently denoted by  $B_\varepsilon$  and  $S_\varepsilon$  respectively.

**Definition 1.3.** Let  $U \subset \mathbb{R}^m$  be an open set and let  $\rho : U \rightarrow \mathbb{R}_{\geq 0}$  be a proper analytic function. We say that the set of  $\rho$ -nonregular points (sometimes called *the Milnor set*, e.g. in [NZ]) of an analytic mapping  $\Psi : U \rightarrow \mathbb{R}^p$  is the set of non-transversality between  $\rho$  and  $\Psi$ , i.e.  $M(\Psi) := \{x \in U \mid \rho \not\llcorner_x \Psi\}$ .

Similarly, the set of  $\rho$ -nonregular points of the mapping  $\frac{\Psi}{\|\Psi\|} : U \setminus V \rightarrow S_1^{p-1}$  is the set:  $M(\frac{\Psi}{\|\Psi\|}) := \text{closure}\{x \in U \setminus V \mid \rho \not\llcorner_x \frac{\Psi}{\|\Psi\|}\}$ .

From the above definition it follows that  $M(\Psi)$  is a relatively closed analytic set containing the singular set  $\text{Sing } \Psi$ . As for  $M(\frac{\Psi}{\|\Psi\|})$ , it is by definition closed but does not necessarily include  $\text{Sing } \Psi$ . We nevertheless have  $M(\frac{\Psi}{\|\Psi\|}) \setminus V \subset M(\Psi) \setminus V$ . In the following we conceive these non-regularity sets as *set germs at the origin*.

It turns out that, in case  $\text{Sing } \psi \subset V$ , the condition<sup>2</sup> “ $\overline{M(\psi) \setminus V} \cap V = \{0\}$ ” insures the existence of the locally trivial fibration  $N \setminus K \xrightarrow{h} B^p \setminus \{0\}$  from Definition 1.1. It

<sup>1</sup>see our Remark 2.1 comparing this statement to another extension formulated by Pichon and Seade in [PS].

<sup>2</sup>condition also used by Massey [Ma] in his study of the local fibration of  $\psi$  near  $V$ .

is implied by the Thom regularity condition at  $V$  and we shall discuss this relation and other criteria in §4. We show by Example 4.1 how to check this condition directly.

In case  $p = 2$  and when  $\psi$  is the pair of functions “the real and the imaginary part” of a holomorphic function germ  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ , Milnor [Mi, §4] had proved that the natural mapping:

$$(2) \quad \frac{f}{|f|} : S_\varepsilon^{2n-1} \setminus K_\varepsilon \rightarrow S_1^1$$

is itself a  $C^\infty$  locally trivial fibration, moreover, that this mapping provides a singular open book decomposition of the odd dimensional spheres  $S_\varepsilon^{2n-1}$  without restriction on the singular locus of  $f$ . We recall that in this setting we have the following inclusion of set-germs at the origin:  $\text{Sing } f \subset V = f^{-1}(0)$ .

In the real analytic setting, Milnor observed that the open book structure (1) may not be induced by the mapping  $\theta = \frac{\psi}{\|\psi\|}$ . He gave an example [Mi, p. 99] of  $\psi$  with isolated singularity,  $m = p = 2$  and  $K = \emptyset$ , showing that the mapping

$$(3) \quad \frac{\psi}{\|\psi\|} : S_\varepsilon^{m-1} \setminus K_\varepsilon \rightarrow S_1^1$$

is not a submersion, hence not a locally trivial fibration.

In case of isolated singularity and  $p = 2$ , several authors obtained sufficient conditions under which (3) is a fibration [Ja1, Ja2, RS, RSV] and provided examples showing that the class of real mapping germs  $\psi$  with isolated singularity satisfying them enlarges the class of holomorphic functions  $f$ . For a more general setting  $\text{Sing } \psi \cap V \subset \{0\}$  and any  $m \geq p \geq 2$  we have given an existence criterion in [AT2]. The following theorem goes beyond this, by allowing singularities in  $V$  too.

**Theorem 1.4.** *Let  $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$  be an analytic mapping germ and suppose that  $\overline{M(\psi) \setminus V} \cap V = \{0\}$ .*

*If  $M(\frac{\psi}{\|\psi\|}) = \emptyset$  then  $(K_\varepsilon, \frac{\psi}{\|\psi\|})$  is an open book structure with singular binding on  $S_\varepsilon^{m-1}$ , independent of  $\varepsilon > 0$  small enough, up to isotopies.*

The assumed condition “ $\overline{M(\psi) \setminus V} \cap V = \{0\}$ ” allows nonisolated singularities since it implies  $M(\psi) = A \cup B$  where  $A \subset V$  and  $B \cap V \subset \{0\}$  and we have seen that  $\text{Sing } \psi \subset M(\psi)$ . Therefore Theorem 1.4 represents a simultaneous extension of [AT2, Theorem 2.2] and of our unpublished [AT1, Proposition 5.3]<sup>3</sup>, where the singular locus is either of type B only or of type A only, respectively. See also Remark 2.2.

We single out here a class of weighted-homogeneous mappings  $\psi$  for which  $(K_\varepsilon, \psi/\|\psi\|)$  is a higher singular open book which is moreover isomorphic to the tube fibration (6). Whereas this behaviour is well-known for holomorphic functions, it is not usual in the category of real analytic mappings. In [PS], it is shown that the mappings of type<sup>4</sup>  $f\bar{g} : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ , where  $f, g : \mathbb{C}^2 \rightarrow \mathbb{C}$  are holomorphic functions such that  $\text{Sing } f\bar{g} \subset V$ , have a Milnor fibration  $f\bar{g}/|f\bar{g}| : S_\varepsilon^3 \setminus K_\varepsilon \rightarrow S^1$  and a “tube fibration” like (4). It

<sup>3</sup>it appears that our result [AT1, Proposition 5.3, Remark 5.4] got lately an expanded proof in [CSS, Theorem 5.3]. The reader could be quite interested to compare [AT1], and more particularly its Section 5, to [CSS] and to the surprising citation about it [CSS, p. 423, lines 5-8].

<sup>4</sup>for more aspects of the germs of type  $f\bar{g}$ , see also [RSV, Se1, Se2]

actually turns out, by using our terminology developed in the present paper, that one has an open book structure with singular binding  $(K_\varepsilon, f\bar{g}/|f\bar{g}|)$ . The proof of [PS, Corollary 1.7] is however not direct, involves the study of the meromorphic germ  $f/g$  and of its link, and holds only in 2 complex variables. It is claimed in *loc.cit.* that it holds for a 2-dimensional space  $(X, 0)$  with isolated singularity in place of the ground space  $(\mathbb{C}^2, 0)$ , but this is presented as a consequence of [PS, Theorem 1.7] which seems to have a gap in the proof; see Remark 2.1 below.

To produce our new class of higher dimensional examples, we use the theory of *mixed functions*, i.e. real analytic mappings  $\mathbb{R}^{2n} \simeq \mathbb{C}^n \rightarrow \mathbb{C} \simeq \mathbb{R}^2$ , recently developed by Mutsuo Oka (see [Oka1, Oka2] and our footnote at §3.2). The necessary definitions are given in §3.

**Theorem 1.5.** *Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  be a mixed polynomial which is radial homogeneous and polar weighted-homogeneous. Then  $(K_\varepsilon, \frac{f}{\|f\|})$  is an open book structure with singular binding on  $S_\varepsilon^{2n-1}$ , independent of  $\varepsilon > 0$  small enough, up to isotopies. Moreover, the Milnor fibration  $\frac{f}{\|f\|} : S_\varepsilon^{2n-1} \setminus K_\varepsilon \rightarrow S^1$  is isomorphic to the tube fibration (6).*

If in the preceding statement one assumes radial weighted-homogeneous instead of homogeneous then the mentioned Milnor fibration on the spheres still exists, a fact which was observed by Oka [Oka1, §5.4] and Cisneros-Molina [CM, Prop. 3.4]. Here we are concerned with the more delicate notion of *open book structure*, hence we want to prove the “good” behaviour near  $K_\varepsilon$  (see the remark after Definition 1.1) which is not an issue in the above mentioned results *loc.cit.* Although the Euler vector field is one of the main ingredients, our proof is different, it is based on our previous theorems and contains parts which are more general, like Proposition 3.1. See §4.1 for remarks concerning the conditions one has to impose near the link, Oka’s result [Oka2, Theorem 52] and Example 4.1.

**EXAMPLE 1.6.**  $f : \mathbb{C}^3 \rightarrow \mathbb{C}$ ,  $f = \bar{x}y^2 + x\bar{z}^2$  has  $\text{Sing } f = \{y = z = 0\} \cup \{x = 0, y = \lambda z \mid \lambda \in S^1\} \subset V$ , hence a *nonisolated* singular locus. It is radial homogeneous of degree 3 and polar weighted-homogeneous of weights  $(3, 2, 1)$  and degree 1, so it verifies the hypotheses of Theorem 1.5, hence its conclusion too.

We produce two new and more elaborated examples in §4, one of them by using a Thom-Sebastiani type statement, Proposition 4.2. The results of §4 are at the origin of a far-reaching extension which will be the object of a subsequent paper [DTY].

**NOTE 1.7.** One may work in a slightly more general setting than Definition 1.1: instead of a manifold  $M$ , let  $M$  be a connected compact real analytic set with  $\text{Sing } M \subset K$ . In this paper  $M$  is the link of  $(\mathbb{R}^m, 0)$ , hence a sphere, but we may also consider a real analytic germ  $(X, 0) \subset (\mathbb{R}^m, 0)$  with connected link  $M$  with respect to some distance function which is not necessarily the Euclidean one, and mappings  $\psi : (X, 0) \rightarrow (\mathbb{R}^p, 0)$  such that  $\text{Sing } X \subset \psi^{-1}(0)$ . Then one has to modify Milnor’s proof §2.1.2 such that the vector field along which one blows the tube to the sphere is tangent to  $X$ . See also Remark 2.1.

Even more generally, if one considers any singular stratified space  $X$  and its link  $M$ , then Milnor’s method reviewed in §2.1 still works, i.e. extends (by using classical technical devices of “radial vector fields”) for stratified transversality and stratified vector fields. This yields what one could call *open book structures with singular pages*.

## 2. PROOF OF THEOREMS 1.2 AND 1.4

**2.1. Revisiting Milnor's method.** In case of mapping germs  $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$ , in Milnor's proof [Mi, p. 97-99, Theorem 11.2] of the existence of a locally trivial fibration (1) one may distinguish two key parts, already pointed out in [AT1, §5]. We explain them below in order to show how they apply to the more general situation displayed in Theorem 1.2.

**2.1.1. Existence of the tube fibration.** Assume that there exists  $\varepsilon_0 > 0$  such that the mapping:

$$(4) \quad \psi| : \bar{B}_\varepsilon^m \cap \psi^{-1}(\bar{B}_\eta^p \setminus \{0\}) \rightarrow \bar{B}_\eta^p \setminus \{0\}$$

is a locally trivial  $C^\infty$ -fibration, for all  $0 < \varepsilon \leq \varepsilon_0$  and  $0 < \eta = \eta(\varepsilon) \ll \varepsilon$ . We shall call it *the tube fibration*.

In the case  $\text{Sing } \psi = \{0\}$  considered by Milnor,  $V$  is transverse to all small enough spheres and therefore any such sphere is also transverse to all nearby fibres, which are moreover non-singular. By Ehresmann theorem, one concludes to the existence of the tube fibration.

**2.1.2. Blowing out the tube to the sphere.** This consists of the “inflation” of the empty tube  $\bar{B}_\varepsilon^m \cap \psi^{-1}(S_\eta^{p-1})$  to the sphere  $S_\varepsilon^{m-1} \setminus K_\varepsilon$ . Milnor explains in [Mi, p. 99] that, given a real analytic mapping  $\psi$ , one may construct<sup>5</sup>, like in [Mi, Lemma 11.3 and §5.10], a nowhere zero  $C^\infty$ -vector field  $v(x)$  on  $\bar{B}_\varepsilon^m \setminus \psi^{-1}(B_\eta^p)$  satisfying the following two conditions:  $\langle x, v(x) \rangle > 0$  and  $\langle \text{grad } \|\psi(x)\|^2, v(x) \rangle > 0$ . The first condition says that  $v(x)$  is transverse to all small spheres and points outwards, and the second conditions says that the mapping  $\|\psi(x)\|^2$  increases along the flow. This vector field may be integrated and produces a diffeomorphism:

$$(5) \quad \gamma : \bar{B}_\varepsilon^m \cap \psi^{-1}(S_\eta^{p-1}) \rightarrow S_\varepsilon^{m-1} \setminus \psi^{-1}(B_\eta^p).$$

This procedure combines the position vector field  $x$  with the gradient vector field  $\text{grad } \|\psi\|^2$  and works if the latter is nowhere zero in the neighbourhood of  $\psi^{-1}(B_\eta^p)$ , if the two vector fields never point in the opposite directions [Mi, Cor. 3.4], and if the empty tube  $\bar{B}_\varepsilon^m \cap \psi^{-1}(S_\eta^{p-1})$  is a manifold with boundary. Milnor uses this construction for holomorphic functions  $f$ , where  $\text{Sing } f \subset V$  and for real mappings with  $\text{Sing } \psi = \{0\}$ . Nevertheless, all these conditions hold locally at 0 whenever  $\text{Sing } \psi \subset V$ .

**2.1.3. Conclusions.** If the tube fibration (4) exists, then its restriction

$$(6) \quad \psi| : \bar{B}_\varepsilon^m \cap \psi^{-1}(S_\eta^{p-1}) \rightarrow S_\eta^{p-1}$$

is a locally trivial  $C^\infty$ -fibration too.

If the inflating procedure works, then the diffeomorphism (5) induces a mapping  $\mu : S_\varepsilon^{m-1} \setminus \psi^{-1}(B_\eta^p) \rightarrow S_\eta^{p-1}$  which is a locally trivial fibration and coincides with  $\psi$  on

---

<sup>5</sup>In an unpublished preprint, Milnor had given an even more explicit construction of such a vector field; a presentation of it may be found in [PS, Theorem 1.3].

$S_\varepsilon^{m-1} \cap \psi^{-1}(S_\eta^{p-1})$ . This may be composed with the mapping  $\frac{s}{\|s\|} : S_\eta^{p-1} \rightarrow S_1^{p-1}$  and yields a locally trivial fibration

$$(7) \quad \mu' : S_\varepsilon^{m-1} \setminus \psi^{-1}(B_\eta^p) \rightarrow S_1^{p-1}$$

**2.2. End of the proof of Theorem 1.2.** The hypothesis  $\overline{M(\psi) \setminus V} \cap V = \{0\}$  is equivalent to the existence of a neighbourhood  $\mathcal{N}$  of  $V \setminus \{0\}$  such that  $M(\psi) \cap \mathcal{N} \setminus V = \emptyset$ . Together with  $\text{Sing } \psi \subset V$ , these imply that for any  $\varepsilon > 0$  small enough, there exists some positive  $\eta \ll \varepsilon$  such that the mapping (4) is a proper submersion. Indeed, the properness follows since the restriction of  $\psi$  to  $\bar{B}_\varepsilon^m \setminus V$  is proper, and “submersion” is a consequence of the condition  $\text{Sing } \psi \subset V$  and of the transversality of the fibres to the boundary  $S_\varepsilon^{m-1} = \partial \bar{B}_\varepsilon^m$ . We may therefore apply Ehresmann’s theorem [Eh, Wo] for manifolds with boundary to conclude to the existence of the locally trivial fibration (4).<sup>6</sup>

To show that the fibration (7) extends to an open book structure on  $S_\varepsilon^{m-1}$  one must produce the application  $\theta$  (cf Definition 1.1). Milnor does not give explicitly the extension of (7) to (1) in case  $\text{Sing } \psi = \{0\}$  since this is an easy matter (see [Mi, Remark, p. 99]: “with a little more effort...”). Indeed, the fibration (7) may be glued along  $S_\varepsilon^{m-1} \cap \psi^{-1}(S_\eta^{p-1})$  to the locally trivial fibration  $S_\varepsilon^{m-1} \cap \psi^{-1}(\bar{B}_\eta^p \setminus \{0\}) \rightarrow \bar{B}_\eta^p \setminus \{0\}$  composed with the mapping  $\frac{s}{\|s\|} : \bar{B}_\eta^p \setminus \{0\} \rightarrow S_1^{p-1}$  since their restrictions to this boundary coincide. This glueing can be done in the  $C^\infty$  category and produces a locally trivial  $C^\infty$ -fibration. We then define  $\theta$  to be the result of the glueing of  $\mu'$  with  $\frac{s}{\|s\|} \circ \psi| : S_\varepsilon^{m-1} \cap \psi^{-1}(\bar{B}_\eta^p \setminus \{0\}) \rightarrow \bar{B}_\eta^p \setminus \{0\} \rightarrow S_1^{p-1}$ , and get that  $(K_\varepsilon, \theta)$  is an open book decomposition of  $S_\varepsilon^{m-1}$ .  $\square$

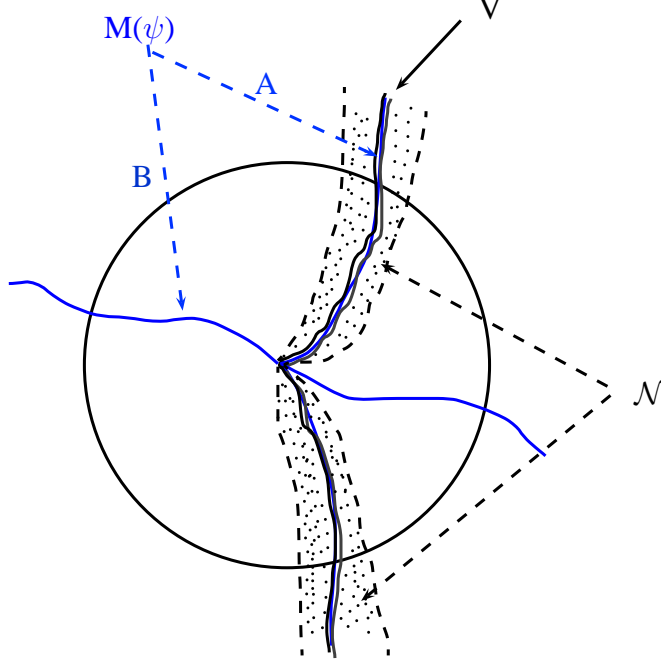
See §4 for criteria concerning the hypothesis  $\overline{M(\psi) \setminus V} \cap V = \{0\}$ .

**REMARK 2.1.** In [PS, Theorem 1.3] the authors state that if  $\text{Sing } \psi \subset V$  and if Thom’s condition holds along  $V$ , then there is an empty tube fibration like (6) and a Milnor fibration out of this tube, which are equivalent. As the authors remark themselves [PS, p. 488 and 492], this is another re-formulation of Milnor’s results which follows from Milnor’s method revisited above. There are the following differences: 1). their hypothesis of Thom regularity implies our condition  $\overline{M(\psi) \setminus V} \cap V = \{0\}$  whereas the converse is presumably not true (see below §4.1 for a discussion of this relation and Example 4.1), and 2). We single out the nice behaviour near the link (hence *inside* the tube) which produces an “open book” structure near the link  $K$ .

Nevertheless, [PS, Theorem 1.3] is stated for a space  $X$  with isolated singularity instead of  $\mathbb{R}^m$ . However Milnor’s original proof as employed in *loc.cit.* does not work in such generality since Milnor’s vector field built up with the position vector field of  $\mathbb{R}^N \supset X$  as ingredient is not necessarily tangent to  $X$ . Apparently the *loc.cit.* use of local charts on  $X \setminus \{0\}$  does not save the situation, since locally the vector field is not globally radial whereas the position vector field is globally defined but not tangent to  $X$ . See also §2.1.2 above and the footnote to it, and see our Note 1.7 on how to work in a more general setting than in [PS].

<sup>6</sup>This observation is also contained in [Ma, Theorem 4.4].

**2.3. Proof of Theorem 1.4.** As explained above at §2.2, the condition  $\overline{M(\psi) \setminus V} \cap V = \{0\}$  is equivalent to the existence of a neighbourhood  $\mathcal{N}$  of  $V \setminus \{0\}$  such that  $M(\psi) \cap \mathcal{N} \setminus V = \emptyset$  and implies the existence of the “empty tube fibration” (6).



This induces a proper submersion  $\frac{\psi}{\|\psi\|} : S_\varepsilon^{m-1} \cap \psi^{-1}(\bar{B}_\eta^p \setminus \{0\}) \rightarrow S_1^{p-1}$  on the manifold with boundary, as in the proof of Theorem 1.2.

On the other hand, the condition that the germ of  $M(\frac{\psi}{\|\psi\|})$  at the origin is empty means that the mapping  $\frac{\psi}{\|\psi\|} : S_\varepsilon^{m-1} \setminus \psi^{-1}(B_\eta^p) \rightarrow S_1^{p-1}$  is a proper submersion for any small enough  $\varepsilon > 0$ . Following the proof of Theorem 1.2, we see that in our situation the particularity is that the mappings to be glued along the boundary  $S_\varepsilon^{m-1} \cap \psi^{-1}(S_\eta^{p-1})$  are actually the same mapping  $\frac{\psi}{\|\psi\|}$  from the both sides, hence the glueing is trivial, and of course smooth, with  $\theta := \frac{\psi}{\|\psi\|}$ . We may therefore conclude as in the proof of Theorem 1.2.  $\square$

**REMARK 2.2.** Let us point out that the “tube” (4) may not be a fibration since our hypotheses allow singularities on fibres close to  $V$ . However, if (6) is a locally trivial fibration with smooth fibres, it follows from the above proof that it is isomorphic to the Milnor fibration:  $\frac{\psi}{\|\psi\|} : S_\varepsilon^{m-1} \setminus K_\varepsilon \rightarrow S_1^{p-1}$ .

### 3. RADIAL AND POLAR WEIGHTED-HOMOGENEOUS MAPPINGS

**3.1. Radial weighted-homogeneous mappings.** Let us consider the  $\mathbb{R}_+$ -action on  $\mathbb{R}^m$ :  $\rho \cdot x = (\rho^{q_1} x_1, \dots, \rho^{q_m} x_m)$  for  $\rho \in \mathbb{R}_+$  and  $q_1, \dots, q_m \in \mathbb{N}^*$  relatively prime positive integers. Let  $\gamma(x) := \sum_{j=1}^m q_j x_j \frac{\partial}{\partial x_j}$  be the corresponding Euler vector field on  $\mathbb{R}^m$ ; we have  $\gamma(x) = 0$  if and only if  $x = 0$ .

We say that the mapping  $\psi$  is *radial weighted-homogeneous* of degree  $d > 0$  if  $\psi(\rho \cdot x) = \rho^d \psi(x)$  for all  $x$  in some neighbourhood of 0.

**Proposition 3.1.** *If  $\psi$  is radial weighted-homogeneous and  $\text{Sing } \psi \subset V$ , then  $M(\frac{\psi}{\|\psi\|}) = \emptyset$ .*

*Proof.* We shall use the following criterion equivalent to  $\{0\} \notin M(\frac{\psi}{\|\psi\|})$  from the proof of [AT2, Theorem 2.2]:  $\exists \varepsilon_0 > 0$  such that  $\text{rank } \Omega_\psi(x) = p$ ,  $\forall x \in B_{\varepsilon_0}^m \setminus V$ , where  $\Omega_\psi(x)$  denotes the  $[(p-1)p/2 + 1] \times m$  matrix defined as:

$$\Omega_\psi(x) := \begin{bmatrix} \omega_{1,2}(x) \\ \vdots \\ \omega_{i,j}(x) \\ \vdots \\ \omega_{p-1,p}(x) \\ x_1, \dots, x_m \end{bmatrix}$$

having on each of the rows the vector  $\omega_{i,j}(x) := \psi_i(x) \text{grad } \psi_j(x) - \psi_j(x) \text{grad } \psi_i(x)$ , for  $i, j = 1, \dots, p$  with  $i < j$ , except of the last row which contains the position vector  $(x_1, \dots, x_m)$ . Observing that  $\langle \gamma(x), \text{grad } \psi_i(x) \rangle = d \cdot \psi_i(x)$  for any  $i$  and any  $x \in B_{\varepsilon_0}^m \setminus V$ , we have:

$$\langle \gamma(x), \omega_{ij}(x) \rangle = d[\psi_i(x)\psi_j(x) - \psi_j(x)\psi_i(x)] = 0,$$

which means that the Euler vector field  $\gamma(x)$  is tangent to the fibres of  $\frac{\psi}{\|\psi\|}$ . We also have:

$$(8) \quad \langle \gamma(x), x \rangle = \sum_i q_i x_i^2 > 0,$$

for  $x \neq 0$ . These show that the position vector  $x$  cannot be orthogonal to the tangent space of the fibres of  $\frac{\psi}{\|\psi\|}$ , which means that the sphere  $S_\varepsilon^{m-1}$  is transverse to the fibres of  $\frac{\psi}{\|\psi\|}$ , for any  $\varepsilon > 0$ .  $\square$

We get the following statement, the proof of which is an immediate consequence of the proof of Theorem 1.4 via Proposition 3.1 and Remark 2.2. It also represents an extension to nonisolated singularities of our previous [AT2, Theorem 3.1].

**Corollary 3.2.** *Let  $\psi$  be radial weighted-homogeneous and let  $\text{Sing } \psi \subset V$ . If the tube fibration (4) exists, then  $(K_\varepsilon, \frac{\psi}{\|\psi\|})$  is a (singular) open book decomposition and the fibration (6) is isomorphic to the Milnor fibration  $\frac{\psi}{\|\psi\|} : S_\varepsilon^{m-1} \setminus K_\varepsilon \rightarrow S_1^{p-1}$ .*  $\square$

**3.2. Mixed functions and polar weighted-homogeneous mappings.** We consider a mixed polynomial<sup>7</sup>  $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu, \mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$  where  $\mathbf{z} = (z_1, \dots, z_n)$ ,  $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$ ,  $\mathbf{z}^\nu = z_1^{\nu_1} \dots z_n^{\nu_n}$ ,  $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \dots \bar{z}_n^{\mu_n}$  for  $\nu = (\nu_1, \dots, \nu_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$  non-negative integer exponents. As a matter of fact, any mixed polynomial is a real polynomial mapping  $\mathbb{R}^{2n} \rightarrow \mathbb{R}^2$ , and reciprocally.

<sup>7</sup>the name “mixed polynomial” was introduced by Oka [Oka1]. The concept has been used before, for instance by A’Campo [AC], then by Ruas, Seade and Verjovsky [RSV], [CM] etc.



One says that  $f$  is *radial weighted-homogeneous* if it is so as a real mapping. Equivalently, there exist positive integers  $q_1, \dots, q_n$  and  $d$  such that  $\gcd(q_1, \dots, q_n) = 1$  and  $\sum_{j=1}^n q_j(\nu_j + \mu_j) = d$ . The corresponding  $\mathbb{R}_+$ -action on  $\mathbb{C}^n$  is:

$$t \cdot (\mathbf{z}, \bar{\mathbf{z}}) = (t^{q_1} z_1, \dots, t^{q_n} z_n, t^{q_1} \bar{z}_1, \dots, t^{q_n} \bar{z}_n), t \in \mathbb{R}_+.$$

After [Oka1] and [CM],  $f$  is called *polar weighted-homogeneous* if there are non-zero integers  $p_1, \dots, p_n$  and  $k$  such that  $\gcd(p_1, \dots, p_n) = 1$  and  $\sum_{j=1}^n p_j(\nu_j - \mu_j) = k$ . The corresponding  $S^1$ -action on  $\mathbb{C}^n$  is:

$$\lambda \cdot (\mathbf{z}, \bar{\mathbf{z}}) = (\lambda^{p_1} z_1, \dots, \lambda^{p_n} z_n, \lambda^{-p_1} \bar{z}_1, \dots, \lambda^{-p_n} \bar{z}_n), \lambda \in S^1.$$

As usual,  $f$  is radial (respectively polar) homogeneous if the weights  $q_i$  (respectively  $p_i$ ) are all equal to 1.

**3.3. Proof of Theorem 1.5.** It was proved in [Oka1] that if  $f$  is radial and polar weighted-homogeneous, then  $\text{Sing } f \subset V$  and there exists a locally trivial fibration  $f|_V : \mathbb{C}^n \setminus V \rightarrow \mathbb{C}^*$ . Let  $c \in \mathbb{C}^*$  be fixed. Since  $f^{-1}(c)$  is algebraic, there exists  $R_c > 0$  such that  $f^{-1}(c) \cap S_R$  for all  $R \geq R_c$ . Moreover, we have:

**Lemma 3.3.**  $f^{-1}(a) \cap S_R$ , for any  $a \in D_{|c|}^*$  and any  $R \geq R_c$ .

*Proof.* If there exist  $a_0 \in D_{|c|}^*$  and  $R_0 \geq R_c$  such that  $f^{-1}(a_0) \not\cap S_{R_0}$ , then there are  $\mathbf{z} = (z_1, \dots, z_n) \in f^{-1}(a) \subset \mathbb{C}^n$ ,  $\gamma \in \mathbb{R}$  and  $\mu \in \mathbb{C}^*$  satisfying the following equation:

$$(9) \quad \gamma z_i = \mu \frac{\overline{\partial f}}{\partial z_i} + \bar{\mu} \frac{\partial f}{\partial \bar{z}_i}$$

for  $1 \leq i \leq n$ . This criterion, proved in [TY], extends Oka's characterisation of the singular locus of  $f$ , cf [Oka1, Prop. 1].

Let  $\lambda_0 \in S^1$ ,  $t_0 \in ]0, 1]$  be the numbers uniquely determined by the equality  $a_0 = t_0 \lambda_0 c$  and set  $\mathbf{w} = (w_1, \dots, w_n) = (\lambda_0^{-p_1} t_0^{-1} z_1, \dots, \lambda_0^{-p_n} t_0^{-1} z_n)$ . Since  $f$  is radial homogeneous and polar weighted-homogeneous, we have the following equations:

$$\begin{aligned} \frac{\partial f}{\partial w_i} &= \lambda_0^{p_i - k} t_0^{1-d} \frac{\partial f}{\partial z_i} \\ \frac{\partial f}{\partial \bar{w}_i} &= \lambda_0^{-p_i - k} t_0^{1-d} \frac{\partial f}{\partial \bar{z}_i} \end{aligned}$$

We take  $\mu_0 = \mu \lambda_0^{-k} t_0^{d-2}$  and  $\gamma_0 = \gamma$ . From the above equations it follows that:

$$\gamma_0 w_i = \mu_0 \frac{\overline{\partial f}}{\partial w_i} + \bar{\mu}_0 \frac{\partial f}{\partial \bar{w}_i}$$

for  $1 \leq i \leq n$ . This contradicts the transversality hypothesis  $f^{-1}(c) \cap S_{\frac{R_0}{t_0}}$  since  $\frac{R_0}{t_0} \geq R_c$ .  $\square$

Lemma 3.3 and the fact that  $\text{Sing } f \subset V$  imply that  $f^{-1}(a)$  is diffeomorphic to  $f^{-1}(a) \cap B_R$ , for any  $a \in D_{|c|}^*$  and any  $R \geq R_c$ .

Let then fix some  $\varepsilon > 0$  and set  $t = \frac{\varepsilon}{R} > 0$ . The  $\mathbb{R}^+$ -action  $t \cdot \mathbf{z} = (tz_1, \dots, tz_n)$  induces the diffeomorphism  $f^{-1}(a) \cap B_R \cong f^{-1}(t^d a) \cap B_\varepsilon$ . Taking  $\delta_\varepsilon = t^d |c|$  it follows that  $f|_V : f^{-1}(D_{\delta_\varepsilon}^*) \cap B_\varepsilon \rightarrow D_{\delta_\varepsilon}^*$  is isomorphic to the locally trivial fibration  $f|_V : \mathbb{C}^n \setminus V \rightarrow \mathbb{C}^*$ , thus

it is a tube fibration. Since the tube fibration exists and since our  $f$  is radial homogeneous, we may apply Corollary 3.2 to conclude our proof.  $\square$

#### 4. CONSTRUCTING MORE EXAMPLES WITH NONISOLATED SINGULARITIES

**4.1. Condition  $\overline{M(\psi)} \setminus \overline{V} \cap V = \{0\}$  and Thom regularity condition.** It is well-known that, if  $V = \psi^{-1}(0)$  may be endowed with a stratification  $\mathcal{S}$  such that in a sufficiently small ball  $B_\epsilon$ , the pair  $(B_\epsilon \setminus V, \mathcal{S})$  satisfies Thom's condition  $(a_\psi)$ , for all  $S \in \mathcal{S}$ , then the stratified transversality of  $V$  to all small enough spheres implies the transversality of the spheres to the nearby fibres in some neighbourhood  $\mathcal{N}$  of  $V \setminus \{0\}$ . This transversality is equivalent to the condition  $\overline{M(\psi)} \setminus \overline{V} \cap V = \{0\}$  and we may repeat the argument in the first part of the proof of Theorem 1.2 to conclude to the existence of the tube fibration (4). This well-known observation<sup>8</sup> may be traced back at least to [HL]<sup>9</sup>, where it is shown that in case of holomorphic function germs  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ , the Thom  $(a_f)$  condition is satisfied<sup>10</sup> along the strata of  $V = f^{-1}(0) \supset \text{Sing } f$  and this implies the existence of the tube fibration (4) due to the condition  $\overline{M(f)} \setminus \overline{V} \cap V = \{0\}$ .

It is conjecturally possible that the condition  $\overline{M(\psi)} \setminus \overline{V} \cap V = \{0\}$  does not imply Thom regularity condition. Let us at least point out two situations where one proves directly the weaker condition  $\overline{M(\psi)} \setminus \overline{V} \cap V = \{0\}$ . One of them is contained in the proof of [Oka2, Lemma 51] in the setting of mixed functions  $f$ , where Oka shows the existence of a tube fibration for a special class of mappings, namely the “super strongly non-degenerate mixed functions”. This is a quite strong condition which allows Oka to prove in [Oka2, Theorem 52] the existence of Milnor fibration on spheres (by extending Milnor's method) and its equivalence to the tube fibration, hence the existence, in our terminology, of an open book structure with singular binding induced by the mapping  $\frac{f}{\|f\|}$ .

Another example of computation is the following one, where the mapping is not a mixed function, hence different from all the precedingly mentioned situations of [Oka2, Theorem 52] or of Theorem 1.5.

**EXAMPLE 4.1.** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $f(x, y, z) = (y^4 - z^2x^2 - x^4, xy)$ . Then  $V(f)$  is the real line  $\{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\}$  and  $\text{Sing } f = V(f)$ .

Let us show that  $\overline{M(f)} \setminus \overline{V} \cap V = \{0\}$ . If this were not true, then, by the Curve Selection Lemma, there are some analytic curves  $x(t)$ ,  $y(t)$ ,  $z(t)$ ,  $a(t)$  and  $b(t)$  defined on a small enough interval  $]0, \epsilon[$  such that  $\lim_{t \rightarrow 0} x(t) = \lim_{t \rightarrow 0} y(t) = 0$ ,  $\lim_{t \rightarrow 0} z(t) = z_0 \neq 0$  and

$$(10) \quad x(t) = a(t)(-4x^3(t) - 2x(t)z^2(t)) + b(t)y(t)$$

$$(11) \quad y(t) = 4a(t)y^3(t) + b(t)x(t)$$

$$(12) \quad z(t) = -2a(t)z(t)x^2(t)$$

<sup>8</sup>see for instance [PS, Theorem 1.3] or [AT1, §5].

<sup>9</sup>see also Hironaka [Hi] and [Lê].

<sup>10</sup>Hamm and Lê attribute this result to F. Pham.

Let us suppose  $x(t) = x_0 t^\beta + \text{h.o.t.}$  where  $x_0 \neq 0$  and  $\beta \in \mathbb{N}$ . From (12) and  $\lim_{t \rightarrow 0} z(t) = z_0$ , we get  $a(t) = -\frac{1}{2x_0^2} t^{-2\beta} + \text{h.o.t.}$  We eliminate  $b(t)$  from (11) and (10) and get:

$$y^2(t) - x^2(t) = a(t)(4y^4(t) + 4x^4(t) + 2x^2(t)z^2(t)).$$

From this equality, since  $\lim_{t \rightarrow 0} 2x^2(t)z^2(t)a(t) = -z_0^2 < 0$  and  $\text{ord}_t(a(t)x^4(t)) = 2\beta > 0$ , and  $\lim_{t \rightarrow 0}(y^2(t) - x^2(t)) = 0$ , we get  $y^4(t) = -1/2x_0^2 z_0^2 t^{2\beta} + \text{h.o.t.}$ , which is a contradiction.

The following Thom-Sebastiani type statement allows one to build further examples.

**Proposition 4.2.** *Consider two mappings in separate variables,  $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$  and  $\phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ , such that  $\text{Sing } \psi \subset V(\psi)$  and  $\text{Sing } \phi \subset V(\phi)$ . Assume that  $\psi$  and  $\phi$  satisfy the Thom regularity condition at  $V(\psi)$  and  $V(\phi)$ , respectively.*

*Then  $\psi + \phi : (\mathbb{R}^m \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  satisfies the Thom regularity condition and there exists a higher open book structure with singular binding  $(K_{\psi+\phi}, \theta)$  on  $S_\varepsilon^{m+p-1}$ , which is independent of  $\varepsilon > 0$  small enough, up to isotopies.*

*If moreover  $\psi$  and  $\phi$  are radial weighted-homogeneous then  $(K_{\psi+\phi}, \frac{\psi+\phi}{\|\psi+\phi\|})$  is a higher open book with singular binding.*

*Proof.* The sum of separate variables mappings which both satisfy the Thom regularity condition has the same property and this follows by a standard check starting from the definition. Since  $\text{Sing } \psi \subset V(\psi)$  and  $\text{Sing } \phi \subset V(\phi)$  it follows that  $\text{Sing } (\psi + \phi) \subset V(\psi + \phi)$ . Then  $\psi + \phi$  verifies the hypotheses of Theorem 1.4 and the first claim follows.

Furthermore, if each mapping is radial weighted-homogeneous, the separate variable sum  $\psi + \phi$  has the same property. Thus one may apply Proposition 3.1 and get the second claim.  $\square$

**EXAMPLE 4.3.** Let  $h : \mathbb{R}^3 \times \mathbb{C}^n \rightarrow \mathbb{R}^2$ ,  $h = f(x, y, z) + g(w_1, \dots, w_n)$ , where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is Example 4.1 and  $g : \mathbb{C}^n \rightarrow \mathbb{C} = \mathbb{R}^2$  is a sum of monomials of the form  $w_i^{k_i}$  or  $\bar{w}_i^{k_i}$  with complex coefficients and its image in  $\mathbb{R}^2$  is  $(\text{Re } g, \text{Im } g)$ . Then  $g$  is radial weighted-homogeneous and has Thom property. As for  $f$ , it is not radial weighted-homogeneous, we have seen above that it satisfies the condition  $\overline{M(f)} \setminus V \cap V = \{0\}$  but we need the Thom regularity condition in order to apply Proposition 4.2 (first claim). We therefore show now that  $f$  satisfies the Thom  $(a_f)$ -regularity condition at  $\text{Sing } f \setminus \{0\}$ . So let us fix some point  $(0, 0, z_0) \in \text{Sing } f = V(f)$ ,  $z_0 \neq 0$ , and choose an analytic curve  $\gamma(t) = (x(t), y(t), z(t))$  with image in  $B_\varepsilon \setminus V(f)$  defined on a small enough interval  $]0, \varepsilon[$  such that  $\lim_{t \rightarrow 0} \gamma(t) = (0, 0, z_0)$ . For any  $t$ , the normal vector field  $v_1(t) = (y(t), x(t), 0)$  to the fibres of  $f$  is orthogonal to the direction  $(0, 0, 1)$  of the line  $V(f)$ . Let us consider the second normal vector field to the fibres of  $f$ , namely  $v_2(t) = (-4x^3(t) - 2x(t)z^2(t), 4y^3(t), -2z(t)x^2(t))$ . If  $x(t) \equiv 0$  then this vector is clearly orthogonal to  $(0, 0, 1)$ , so let us assume that  $\text{ord}_t x(t) > 0$ . Dividing  $v_2(t)$  by  $x(t)$  and taking the limit as  $t \rightarrow 0$  we get  $(-2z_0^2, *, 0)$ , which implies that  $\langle \lim_{t \rightarrow 0} \frac{v_2(t)}{\|v_2(t)\|}, (0, 0, 1) \rangle = 0$ . These show that the tangent space to the line  $\text{Sing } f$  is contained in the limit of the tangent spaces to the fibres of  $f$  and hence prove that  $f$  satisfies the Thom  $(a_f)$ -regularity condition at  $V(f) \setminus \{0\}$ .

## REFERENCES

- [AC] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*. Indag. Math. 35 (1973), 113–118.
- [Ar1] R.N. Araújo dos Santos, *Uniform (M)-condition and strong Milnor fibrations*, Singularities II, 189–198, Contemp. Math., 475, Amer. Math. Soc., Providence, RI, 2008.
- [Ar2] R.N. Araújo dos Santos, *Equivalence of real Milnor fibrations for quasi-homogeneous singularities*, Rocky Mountain J. Math., to appear.
- [AT1] R.N. Araújo dos Santos, M. Tibăr, *Real mapping germs and higher open books*, submitted to LMS, December 7, 2007. preprint arXiv:0801.3328.
- [AT2] R.N. Araújo dos Santos, M. Tibăr, *Real mapping germs and higher open book structures*, Geometriae Dedicata 147 (2010), 177–185.
- [CM] J.L. Cisneros-Molina, *Join theorem for polar weighted-homogeneous singularities*. Singularities II, 43–59, Contemp. Math., 475, Amer. Math. Soc., Providence, RI, 2008.
- [CSS] J.L. Cisneros-Molina, J. Seade, J. Snoussi, *Milnor fibrations and d-regularity for real analytic singularities*. Internat. J. Math. 21 (2010), no. 4, 419–434.
- [DTY] R. Dias, M. Tibăr, Chen Ying, *Sebastiani-Thom type theorems and Thom condition*, manuscript.
- [Eh] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*. Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 29–55. Georges Thone, Liège; Masson et Cie., Paris, 1951.
- [HL] H. Hamm, Lê D.T., *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. (4) 6 (1973), 317–355.
- [Hi] H. Hironaka, *Stratification and flatness*. Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 199–265. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
- [Ja1] A. Jacquemard, *Thèse 3ème cycle Université de Dijon*, 1982.
- [Ja2] A. Jacquemard, *Fibrations de Milnor pour des applications réelles*, Boll. Un. Mat. Ital. B (7) 3 (1989), no. 3, 591–600.
- [Lê] Lê D.T., *Some remarks on the relative monodromy*, in: Real and Complex Singularities, Oslo 1976, Sijthoff en Norhoff, Alphen a.d. Rijn 1977, pp. 397–403.
- [Lo] E.J.N. Looijenga, *A note on polynomial isolated singularities*. Indag. Math. 33 (1971), 418–421.
- [Ma] D. Massey, *Real Analytic Milnor Fibrations and a Strong Lojasiewicz Inequality*, arXiv:math/0703613v5.
- [Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton University Press, 1968.
- [NZ] A. Némethi, A. Zaharia, *Milnor fibration at infinity*. Indag. Math. 3.(1992) , 323–335.
- [Oka1] M. Oka, *Topology of polar weighted-homogeneous hypersurfaces*, Kodai Math. J. 31 (2008), no. 2, 163–182.
- [Oka2] M. Oka, *Non degenerate mixed functions*, Kodai Math. J. 33 (2010), no. 1, 1–62.
- [PS] A. Pichon, J. Seade, *Fibred multilinks and singularities  $f\bar{g}$* . Math. Ann. 342 (2008), no. 3, 487–514.
- [RS] M.A.S. Ruas and R.N. Araújo dos Santos, *Real Milnor Fibrations and (C)–regularity*, Manuscripta Math., 117, (2005), no. 2, 207–218.
- [RSV] M.A.S. Ruas, J. Seade and A. Verjovsky, *On Real Singularities with a Milnor Fibration*, in: Trends in Singularities, 191–213, Trends Math., Birkhäuser, Basel, 2002.
- [Se1] J. Seade, *On the topology of isolated singularities in analytic spaces*. Progress in Mathematics, 241. Birkhuser Verlag, Basel, 2006. xiv+238 pp.
- [Se2] J. Seade, *On Milnor's fibration theorem for real and complex singularities*. in: Singularities in geometry and topology, 127–158, World Sci. Publ., Hackensack, NJ, 2007.
- [Th] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. 75 (1969), 249–312.

- [Ti1] M. Tibăr, *Regularity at infinity of real and complex polynomial mappings*, Singularity Theory, The C.T.C Wall Anniversary Volume, LMS Lecture Notes Series 263 (1999), 249–264. Cambridge University Press.
- [Ti2] M. Tibăr, *Polynomials and vanishing cycles*. Cambridge Tracts in Mathematics, 170. Cambridge University Press, Cambridge, 2007. xii+253 pp.
- [TY] M. Tibăr, Chen Ying, *Bifurcation values of mixed polynomials*, arXiv:1011.4884v1, submitted August 2010.
- [Wi] H.E. Winkelnkemper, *Manifolds as open books*, Bull. Amer. Math. Soc. 79 (1973), 45–51.
- [Wo] J.A. Wolf, *Differentiable fibre spaces and mappings compatible with Riemannian metrics*. Michigan Math. J. 11 (1964), 65–70.

*E-mail address:* `tibar@math.univ-lille1.fr`

*E-mail address:* `Ying.Chen@math.univ-lille1.fr`

MATHÉMATIQUES, UMR-CNRS 8524, UNIVERSITÉ LILLE 1, 59655 VILLENEUVE D’ASCQ, FRANCE.

*E-mail address:* `rnonato@icmc.usp.br`

ICMC, UNIVERSIDADE DE SÃO PAULO, AV. TRABALHADOR SÃO-CARLENSE, 400 - CP Box 668, 13560-970 SÃO CARLOS, SÃO PAULO - BRAZIL